ON THE MOTION OF A GYROSCOPIC PENDULUM WITH RANDOM DISPLACEMENTS OF ITS POINT OF SUSPENSION

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1. The motion of a gyroscopic pendulum whose point of suspension is subjected to the horizontal acceleration components A_ξ and A_η has been studied by many authors. It is well known [1] that if the nutational motion and damping are not considered, then for small angular defections of the pendulum axis from the vertical, α and β, this problem reduces to the solution of a system of linear first order differential equations

$$\dot{\alpha} - kg\beta = -kA_\eta, \qquad \dot{\beta} + kg\alpha = kA_\xi, \qquad k = \frac{ml}{H} \tag{1.1}$$

where g is the acceleration due to gravity, m is the mass of the gyro housing and the rotor, and H is the kinetic moment of the rotor of the gyroscope.

In the usual statement of the problem A_ξ and A_η are assumed to be given functions of time, and the analysis of the system (1.1) does not present any fundamental difficulties. In some problems, however, in addition to the horizontal components of acceleration of the suspension point, the vertical component of acceleration A_ζ can also produce a significant effect, and all three acceleration components are random functions of time.

An example of such a problem is the study of the motion of a gyroscopic pendulum on a ship, where the presence of heaving causes a random displacement of the pendulum support. A similar problem arises during the study of the behavior of a gyroscopic pendulum carried by an airplane, as well as in a number of other cases.

For problems of such type, the following system of equations of motion can be written considering the assumptions made above

$$\dot{\alpha} - kg\left(1 + \frac{1}{g} A_\zeta\right)\beta = -kA_\eta, \qquad \dot{\beta} + kg\left(1 + \frac{1}{g} A_\zeta\right)\alpha = kA_\xi \tag{1.2}$$

where A_ξ, A_η and A_ζ, and consequently $\alpha(t)$ and $\beta(t)$, are random functions of time.

With the presence of damping, which can occur as a result of air resistance or forces of viscous drag at the support axes, the system (1.2) changes into the following system of equations

$$\alpha - kg\left(1 + \frac{1}{g} A_\zeta\right)\beta - n\dot{\beta} = -kA_\eta, \quad \dot{\beta} + kg\left(1 + \frac{1}{g} A_\zeta\right)\alpha + n\dot{\alpha} = kA_\xi \tag{1.3}$$

Here n is the ratio of the damping coefficient (we assume it to be the same for both axes of the universal support) to the kinetic moment of the gyroscope rotor H.

When we introduce instead of the real functions $\alpha(t)$ and $\beta(t)$ the complex function $\gamma(t)$, the above system can be written in terms of a single equation

$$\dot{\gamma}(t) + ik_1 g\,[1 + Y(t)]\,\gamma(t) = V_1(t) + iW_1(t) \qquad (\gamma = \alpha + i\beta) \tag{1.4}$$

Here

$$k_1 g = \omega_1 (1 - in), \qquad \omega_1 = \frac{kg}{1 + n^2} = \frac{\omega}{1 + n^2} \tag{1.5}$$

$$V_1(t) = \frac{1}{1 + n^2}[V(t) + nW(t)], \qquad W_1(t) = \frac{1}{1 + n^2}[W(t) - nV(t)] \tag{1.6}$$

$$V(t) = -kA_\eta(t), \qquad W(t) = kA_\xi(t), \qquad Y(t) = \frac{1}{g} A_\zeta(t) \tag{1.7}$$

Without damping, the solution of the problem can be obtained from the solution of Equation (1.4) if $n = 0$ is substituted. The function $\gamma(t)$ is random, and consequently, the solution of Equation (1.4) reduces to the problem of determining all laws of distribution of the ordinates of this function. Even with conditions of normality on the random processes $Y(t)$, $V(t)$ and $W(t)$ this problem is very complicated. For the purpose of applications, however, it is sufficient to know only the mathematical expectation and dispersion of the angular deflections $\alpha(t)$ and $\beta(t)$. It will be shown later that the moments of these functions can be determined in general form for any system of random functions $Y(t)$, $V(t)$ and $W(t)$, and in the case when these functions are normal one can obtain simple computational formulas.

Since $|\gamma(t)| = \sqrt{(\alpha^2 + \beta^2)}$ gives the deviation of the axis of the gyropendulum from the vertical, the following quantities are of direct interest

$$\boldsymbol{M}[\alpha(t)] = \operatorname{Re} \boldsymbol{M}[\gamma(t)], \qquad \boldsymbol{M}[\beta(t)] = \operatorname{Im} \boldsymbol{M}[\gamma(t)] \tag{1.8}$$

$$\boldsymbol{D}[|\gamma(t)|] = \boldsymbol{M}[|\gamma(t)|^2] - \{\boldsymbol{M}[|\gamma(t)|]\}^2 \tag{1.9}$$

and also the dispersions of $\alpha(t)$ and $\beta(t)$, which can be determined from the formulas

$$\begin{aligned} \boldsymbol{D}[\alpha(t)] &= \tfrac{1}{2}\boldsymbol{M}[|\gamma(t)|^2] + \tfrac{1}{2}\operatorname{Re}\boldsymbol{M}[\gamma^2(t)] - \{\boldsymbol{M}[\alpha(t)]\}^2 \\ \boldsymbol{D}[\beta(t)] &= \tfrac{1}{2}\boldsymbol{M}[|\gamma(t)|^2] - \tfrac{1}{2}\operatorname{Re}\boldsymbol{M}[\gamma^2(t)] - \{\boldsymbol{M}[\beta(t)]\}^2 \end{aligned} \tag{1.10}$$

2. Lower case latin letters will denote the mathematical expectation of the random functions denoted by corresponding capital latin letters. Let

$$\gamma(t) = \delta(t) + c, \qquad c = \frac{v_1(t) + iw_1(t)}{ik_1 g\,[1 + y(t)]} \tag{2.1}$$

where $\delta(t)$ satisfies the differential equation

$$\dot{\delta}(t) + igk_1[1 + Y(t)]\,\delta(t) = V_1^\circ(t) + iW_1^\circ(t) \tag{2.2}$$

$$V_1^\circ(t) = \frac{1}{1+n^2}[V^\circ(t) + nW^\circ(t)], \qquad W_1^\circ(t) = \frac{1}{1+n^2}[W^\circ(t) - nV^\circ(t)] \tag{2.3}$$

while the functions $V^\circ(t)$ and $W^\circ(t)$ are determined by the equations

$$\begin{aligned} V^\circ(t) &= V(t) - v(t) - \frac{v(t)}{1+y(t)}[Y(t) - y(t)] \\ W^\circ(t) &= W(t) - w(t) - \frac{w(t)}{1+y(t)}[Y(t) - y(t)] \end{aligned} \tag{2.4}$$

Equation (2.2) does not differ in appearance from the original Equation (1.4), however, at its right-hand side, it contains functions which have a zero mathematical expectation. Let us represent the random function $\delta(t)$ in the form of a sum

$$\delta(t) = \delta_0(t) + \delta_1(t) \tag{2.5}$$

where $\delta_0(t)$ is the solution of the homogeneous equation corresponding to Equation (1.4) and the given initial conditions α_0, β_0 for $\alpha(t)$ and $\beta(t)$; $\delta_1(t)$ satisfies (1.5) and zero initial conditions. Of course,

$$\boldsymbol{M}[\delta(t)] = \boldsymbol{M}[\delta_0(t)] + \boldsymbol{M}[\delta_1(t)] \tag{2.6}$$

Considering (2.1) and using the expression

$$Z(t_1) = \int_{t_1}^{t} Y(t_2)\,dt_2 \tag{2.7}$$

we obtain

$$\delta_0(t) = (\alpha_0 + i\beta_0 - c)\exp\{-ik_1 g[t + Z(0)]\} \tag{2.8}$$

$$\delta_1(t) = \int_0^t \exp\{-ik_1 g(t - t_1) - ik_1 g Z(t_1)\}[V_1{}^\circ(t_1) + iW_1{}^\circ(t_1)]\,dt_1 \tag{2.9}$$

After computing the mathematical expectation of both parts of (2.7) we obtain

$$M[\delta_1(t)] = \int_0^t e^{-ik_1 g(t-t_1)} M\{e^{-ik_1 g Z(t_1)}[V_1{}^\circ(t_1) + iW_1{}^\circ(t_1)]\}\,dt_1 \tag{2.10}$$

Let us denote the characteristic functions of the systems of the random quantities $Z(t)$, $V_1{}^\circ(t)$, and consequently, $Z(t_1)$, $W_1{}^\circ(t_1)$ by $E(u_1, u_2)$ and $E(u_1, u_3)$. Then we shall have

$$M\{e^{-ik_1 g Z(t_1)}[V_1{}^\circ(t_1) + iW_1{}^\circ(t_1)]\} =$$

$$= \frac{1}{i}\frac{\partial}{\partial u_2}E(u_1, u_2) + \frac{\partial}{\partial u_3}E(u_1, u_3) \quad \text{for} \begin{cases} u_1 = -k_1 g \\ u_2 = u_3 = 0 \end{cases} \tag{2.11}$$

In our further analyses, let us restrict ourselves only to the normal, stationary, random functions $Y(t)$, $V^\circ(t)$ and $W^\circ(t)$. In this case the function $Z(t)$ will also be normal. Since the characteristic function $E(u_1, \ldots, u_n)$ of the system of normal random quantities $X_1, \ldots, X_n$ is uniquely expressed in terms of the elements of the correlation matrix $\| k_{jl} \|$ of this system, and their mathematical expectation x_j is expressed by formula [2]

$$E(u_1, \ldots u_n) = \exp\left(-\frac{1}{2}\sum_{j,l=1}^{n} k_{jl}u_j u_l + i\sum_{j=1}^{n} u_j x_j\right) \tag{2.12}$$

we obtain instead of (2.10)

$$M[\delta_1(t)] = \frac{(1 - in)^2\,\omega_1}{1 + n^2} \times$$

$$\times \int_0^t \exp\left[-ik_1 g(1 + \eta)(t - t_1) - \frac{1}{2}k_1{}^2 g^2 k_{11}\right](k_{13} - ik_{12})\,dt_1 \tag{2.13}$$

Here

$$k_{11} = \boldsymbol{D}\,[Z(t_1)],\ \ k_{12} = \boldsymbol{M}\,\{[Z(t_1) - z(t_1)]\,V_1^{\circ}(t_1)\}$$
$$k_{13} = \boldsymbol{M}\,\{[Z(t_1) - z(t_1)]\,W_1^{\circ}(t_1)\} \tag{2.14}$$

When we denote by $K_y(\tau)$ the correlation function of $Y(t)$ and by $R_{vy}(\tau)$ and $R_{wy}(\tau)$. the correlation functions of the coupling of $V^{\circ}(t)$ with $Y(t)$ and consequently of $W^{\circ}(t)$ with $Y(t)$, and let

$$f(\tau) = 2\int_0^{\tau}(\tau - \tau_1)\,\boldsymbol{K}_y(\tau_1)\,d\tau_1, \quad \varphi_1(\tau) = \int_0^{\tau} R_{vy}(\tau_1)\,d\tau_1, \quad \varphi_2(\tau) = \int_0^{\tau} R_{wy}(\tau_1)\,d\tau_1 \tag{2.15}$$

the moments (2.14) become

$$k_{11} = f(t - t_1), \qquad k_{12} = \frac{1}{1+n^2}[\varphi_1(t - t_1) - n\,\varphi_2(t - t_1)]$$
$$k_{13} = \frac{1}{1+n^2}[\varphi_2(t - t_1) + n\varphi_1(t - t_1)] \tag{2.16}$$

By substituting (2.16) into (2.13) we can separate the real part from the imaginary part:

$$\begin{aligned}\boldsymbol{M}[\delta_1(t)] = \frac{\omega_1}{1+n^2}\int_0^t \exp[L(\tau)]\,\{[\varphi_2(\tau) - n\varphi_1(\tau)]\cos N(\tau) + \\ + [\varphi_1(\tau) + n\varphi_2(\tau)]\sin N(\tau)\}\,d\tau + \\ + \frac{i\omega_1}{1+n^2}\int_0^t \exp[L(\tau)]\,\{[\varphi_2(\tau) - n\varphi_1(\tau)]\sin N(\tau) + \\ + [\varphi_1(\tau) + n\varphi_2(\tau)]\cos N(\tau)\}\,d\tau\end{aligned} \tag{2.17}$$

$$L(\tau) = -\,\omega_1\Big[n(1+y)\,\tau + \frac{1}{2}\omega_1 f(\tau)\Big], \qquad N(\tau) = \omega_1[(1+y)\,\tau - \omega_1 f(\tau)]$$

The mathematical expectation $M[\delta_0(\tau)]$ can be expressed in terms of the characteristic function $E(u)$ of the quantity

$$\boldsymbol{M}[\delta_0(t)] = \{\boldsymbol{M}[\alpha_0] + i\boldsymbol{M}[\beta_0] - c\}\,e^{-ik_1gt}\,\boldsymbol{E}(u) \ \text{ when } u = -\,k_1g \tag{2.18}$$

After expressing $E(u)$ explicitely in terms of the moments we obtain

$$\boldsymbol{M}[\delta_0(t)] = \exp\Big\{-\,\omega_1\Big[n(1+y)\,t + \frac{1}{2}\,\omega_1(1-n^2)\,f(t)\Big]\Big\} \times$$
$$\times\,\Big\{\Big[\boldsymbol{M}[\alpha_0] - \frac{w}{\omega(1+y)}\Big]\cos\omega_1[(1+y)\,t - n\omega_1 f(t)] +$$

$$+\left[\boldsymbol{M}[\beta_0]+\frac{v}{\omega(1+y)}\right]\sin\omega_1[(1+y)t-n\omega_1 f(t)]\Big\}+$$
$$+i\exp\left\{-\omega_1\left[n(1+y)t+\frac{1}{2}\omega_1(1-n^2)f(t)\right]\right\}\times$$
$$\times\left\{\left[\boldsymbol{M}[\beta_0]+\frac{v}{\omega(1+y)}\right]\cos\omega_1[(1+y)t-n\omega_1 f(t)]-\right.$$
$$\left.-\left[\boldsymbol{M}[\alpha_0]-\frac{w}{\omega(1+y)}\right]\sin\omega_1[(1+y)t-n\omega_1 f(t)]\right\} \qquad (2.19)$$

Formulas (2.9), (2.17) and (2.19) completely determine the mathematical expectation of the angles $\alpha(t)$ and $\beta(t)$.

For the determination of $M[|\delta_1(t)|^2]$ we multiply (2.9) by its complex conjugate and obtain

$$|\delta_1(t)|^2=\int_0^t\int_0^t\exp\{-i\omega_1[(1-in)(t-t_1)-(1+in)(t-t_2)+(1-in)Z(t_1)-$$
$$-(1+in)Z(t_2)]\}\,[V_1^\circ(t_1)+iW_1^\circ(t_1)]\,[V_1^\circ(t_2)-iW_1^\circ(t_2)]\,dt_1dt_2 \qquad (2.20)$$

Let us study the system of normal random quantities $Z(t_1)$, $Z(t_2)$, $V^\circ(t_1)$, $V^\circ(t_2)$, $W^\circ(t_1)$, $W^\circ(t_2)$ which we number in the order of their appearance. We shall also identify the arguments u_r $(r = 1, \ldots, 6)$ of the characteristic function $\boldsymbol{E}$ by the same numbers as those of the random quantities. Then, we obtain for the mathematical expectation of (2.20)

$$\boldsymbol{M}[|\delta_1(t)|^2]=\frac{1}{1+n^3}\int_0^t\int_0^t\exp\{-i\omega_1[(1-in)(t-t_1)-(1+in)(t-t_2)]\}\times$$
$$\times\left\{-\frac{\partial^2}{\partial u_3\partial u_4}\boldsymbol{E}(u_1,u_2,u_3,u_4)-\frac{\partial}{\partial u_5\partial u_6}\boldsymbol{E}(u_1,u_2,u_5,u_6)+\right.$$
$$\left.+i\frac{\partial^2}{\partial u_3\partial u_6}\boldsymbol{E}(u_1,u_2,u_3,u_6)-i\frac{\partial^2}{\partial u_4\partial u_5}\boldsymbol{E}(u_1,u_2,u_4,u_5)\right\}dt_1dt_2$$
$$\text{when } u_1=-\omega_1(1-in),\ u_2=\omega_1(1+in),\ u_3=u_4=u_5=u_6=0 \qquad (2.21)$$

When we also express here the characteristic functions by the elements of the correlation matrix $\|k_{jl}\|$ of the system of random quantities under study and assume that $M[Z(t_1)] = (t - t_1)y$ we obtain

$$\boldsymbol{M}[|\delta_1(t)|^2]=\frac{\omega_1^2}{1+n^2}\int_0^t\int_0^t\exp\{-i\omega_1[(1-in)(t-t_1)-(1+in)(t-t_2)]-$$
$$-\tfrac{1}{2}\omega_1^2[(1-in)^2k_{11}+(1+in)^2k_{22}-2(1+n^2)k_{12}]-$$
$$-i\omega_1 y[(t_2-t_1)-in(2t-t_1-t_2)]\}\{-[(k_{31}-k_{32})-in(k_{31}+k_{32})]\times$$

$$\times [(k_{41} - k_{42}) - in(k_{41} + k_{42})] - [(k_{51} - k_{52}) - in(k_{51} + k_{52})] \times \quad (2.22)$$

$$\times [(k_{61} - k_{62}) - in(k_{61} + k_{62})] + i[(k_{31} - k_{32}) - in(k_{31} + k_{32})] \times$$

$$\times [(k_{61} - k_{62}) - in(k_{61} + k_{62})] - i[(k_{41} - k_{42}) - in(k_{41} + k_{42})] \times$$

$$\times [(k_{51} - k_{52}) - in(k_{51} + k_{52})] + \omega_1^{-2}(k_{34} + k_{56} - ik_{36} + ik_{45})\} dt_1, dt_2$$

When using (2.15) and denoting the correlation functions of $V^{\circ}(t)$ and $W^{\circ}(t)$ by $K_v(\tau)$ and $K_w(\tau)$ the last equation can be written in the form

$$\boldsymbol{M}[|\delta_1(t)|^2] = \int_0^t\int_0^t e^{-D(\tau_1,\tau_2)}\{A(\tau_1,\tau_2)\cos[C(\tau_1,\tau_2)] - B(\tau_1,\tau_2)\sin[C(\tau_1,\tau_2)]\}\,d\tau_1 d\tau_2 \quad (2.23)$$

where

$$A(\tau_1,\tau_2) = \frac{\omega_1^2}{1+n^2}[\varphi_1(\tau_2-\tau_1)\varphi_1(\tau_1-\tau_2) + \varphi_2(\tau_2-\tau_1)\varphi_2(\tau_1-\tau_2)] +$$

$$+ \frac{2n\omega_1^2}{1+n^2}[\varphi_1(\tau_1-\tau_2)\varphi_2(\tau_2) + \varphi_1(\tau_2-\tau_1)\varphi_2(\tau_1) - \varphi_2(\tau_2-\tau_1)\varphi_1(\tau_1) -$$

$$- \varphi_2(\tau_1-\tau_2)\varphi_1(\tau_2)] + \frac{2n^2\omega_1^2}{1+n^2}[2\varphi_1(\tau_1)\varphi_1(\tau_2) + 2\varphi_2(\tau_1)\varphi_2(\tau_2) -$$

$$- \varphi_1(\tau_1-\tau_2)\varphi_1(\tau_2) - \varphi_1(\tau_2-\tau_1)\varphi_1(\tau_1) - \varphi_2(\tau_2-\tau_1)\varphi_2(\tau_1) -$$

$$- \varphi_2(\tau_1-\tau_2)\varphi_2(\tau_2)] + \frac{1}{1+n^2}[K_v(\tau_2-\tau_1) + K_w(\tau_2-\tau_1)] \quad (2.24)$$

$$B(\tau_1,\tau_2) = \frac{2n\omega_1^2}{1+n^2}\{[2\varphi_1(\tau_1-\tau_2)\varphi_1(\tau_2) + 2\varphi_2(\tau_1-\tau_2)\varphi_2(\tau_2) +$$

$$+ \varphi_1(\tau_2-\tau_1)\varphi_2(\tau_1-\tau_2)] - [2\varphi_1(\tau_2-\tau_1)\varphi_1(\tau_1) + 2\varphi_2(\tau_2-\tau_1)\varphi_2(\tau_1) +$$

$$+ \varphi_1(\tau_1-\tau_2)\varphi_2(\tau_2-\tau_1)] + n[2\varphi_1(\tau_2) - \varphi_1(\tau_2-\tau_1)][2\varphi_2(\tau_1) - \varphi_2(\tau_1-\tau_2)] -$$

$$- n[2\varphi_1(\tau_1) - \varphi_1(\tau_1-\tau_2)][2\varphi_2(\tau_2) - \varphi_2(\tau_2-\tau_1)]\} \quad (2.25)$$

$$C(\tau_1,\tau_2) = \omega_1[(\tau_2-\tau_1) - n\omega_1 f(\tau_2) + n\omega_1 f(\tau_1) + (\tau_2-\tau_1)y] \quad (2.26)$$

$$D(\tau_1,\tau_2) = \frac{1}{2}\omega_1^2[(1+n^2)f(\tau_2-\tau_1) - 2n^2 f(\tau_1) - 2n^2 f(\tau_2)] + \omega_1 n(\tau_1+\tau_2)(1+y) \quad (2.27)$$

From (2.5) it follows that

$$\boldsymbol{M}[|\delta(t)|^2] = \boldsymbol{M}[|\delta_0(t)|] + \boldsymbol{M}[|\delta_1(t)|^2] + 2\,\mathrm{Re}\,\boldsymbol{M}[\delta_0(t)\delta_1(t)] \quad (2.28)$$

The mathematical expectations entering the last equation can also be found by the same method as $M[|\delta_1(t)|^2]$. If we assume for simplicity that α_0 and β_0 are mutually independent, have zero mathematical expectations, and do not depend on $Y(t)$, $V^{\circ}(t)$ and $W^{\circ}(t)$, we obtain

$$\boldsymbol{M}[|\delta_0(t)|^2] = \{\boldsymbol{D}[\alpha_0] + \boldsymbol{D}[\beta_0]\} \exp[-2n\omega_1 t + 2n^2\omega_1^2 f(t)]$$
$$\operatorname{Re} \boldsymbol{M}[\delta_0(t)\,\delta_1(t)] = 0 \tag{2.29}$$

Analogously to Formula (2.23) it follows from (2.9) that

$$\operatorname{Re} \boldsymbol{M}[\delta_1^2(t)] = \int_0^t\int_0^t e^{-D_1(\tau_1,\tau_2)} \{[(1-n^2) A_1(\tau_1,\tau_2) + 2nB_1(\tau_1,\tau_2)] \cos C_1(\tau_1,\tau_2) +$$
$$+ [(1-n^2) B_1(\tau_1,\tau_2) - 2nA_1(\tau_1,\tau_2)] \sin C_1(\tau_1,\tau_2)\}\, d\tau_1 d\tau_2 \tag{2.30}$$

where

$$A_1(\tau_1,\tau_2) = \boldsymbol{K}_w(\tau_2-\tau_1) - \boldsymbol{K}_v(\tau_2-\tau_1) + \omega_1^2(1-n^2) \times$$
$$\times [\Phi_{2,2}(\tau_1,\tau_2) - \Phi_{1,1}(\tau_1,\tau_2)] + 2\omega_1^2 n [\Phi_{1,2}(\tau_1,\tau_2) + \Phi_{1,2}(\tau_2,\tau_1)] \tag{2.31}$$

$$B_1(\tau_1,\tau_2) = -R_{vw}(\tau_1-\tau_2) - R_{vw}(\tau_2-\tau_1) - (1-n^2)\omega_1 \times$$
$$\times [\Phi_{1,2}(\tau_1,\tau_2) + \Phi_{1,2}(\tau_2,\tau_1)] - 2n\omega_1^2 [\Phi_{2,2}(\tau_1,\tau_2) + \Phi_{1,1}(\tau_1,\tau_2)] \tag{2.32}$$

$$C_1(\tau_1,\tau_2) = \omega_1\{(\tau_1+\tau_2) - n\omega_1[3f(\tau_1) + 3f(\tau_2) - f(\tau_2-\tau_1)]\} \tag{2.33}$$

$$D_1(\tau_1,\tau_2) = \omega_1\{n(\tau_1+\tau_2) + \frac{1}{2}(1-n^2)\omega_1[3f(\tau_1) + 3f(\tau_2) - f(\tau_2-\tau_1)]\}$$
$$\Phi_{j,l}(\tau_p,\tau_q) = [2\varphi_j(\tau_p) - \varphi_j(\tau_p-\tau_q)][2\varphi_l(\tau_q) - \varphi_l(\tau_q-\tau_p)] \tag{2.34}$$
$$(j, l, p, q = 1, 2) \tag{2.35}$$

and $R_{vw}(\tau)$ is the correlation function of the coupling of $V^{\circ}(t)$ and $W^{\circ}(t)$. In such a fashion we obtain from (2.8)

$$\operatorname{Re} \boldsymbol{M}[\delta_0^2(t)] = \{\boldsymbol{D}[\alpha_0] - \boldsymbol{D}[\beta_0] + c^2\} \exp[-2n\omega_1 t - 2(1-n^2)\omega_1^2 f(t)] \times$$
$$\times \cos\{2\omega_1[t - 2\omega_1 n f(t)]\} \tag{2.36}$$

Finally, considering the assumptions on the initial conditions, we obtain

$$\operatorname{Re} \boldsymbol{M}[\delta^2(t)] = \operatorname{Re} \boldsymbol{M}[\delta_0^2(t)] + \operatorname{Re} \boldsymbol{M}[\delta_1^2(t)] \tag{2.37}$$

i.e. we will have all necessary formulas for the calculation of the dispersions of $\alpha(t)$ and $\beta(t)$, determined by the equalities (1.10).

3. Let us now analyze the obtained results. At first, let us study the motion of a gyropendulum without any damping. After substituting $n = 0$ into (2.17) and (2.19) and assuming that in that case $\omega_1 = \omega$, we obtain for the mathematical expectations of the angular deflections of the axis

$$\boldsymbol{M}[\alpha(t)] = F(t)\left\{\left[\boldsymbol{M}[\alpha_0] - \frac{w}{\omega(1+y)}\right]\cos[\omega(1+y)t] + \right.$$

$$+\left.\left[\boldsymbol{M}[\beta_0] + \frac{v}{\omega(1+y)}\right]\sin[\omega(1+y)t]\right\} + \omega\int_0^t F(\tau)\varphi_2(\tau)\cos[\omega(1+y)\tau]\,d\tau +$$

$$+\omega\int_0^t F(\tau)\varphi_1(\tau)\sin[\omega(1+y)\tau]\,d\tau \qquad (3.1)$$

$$\boldsymbol{M}[\beta(t)] = F(t)\left\{\left[\boldsymbol{M}[\beta_0] + \frac{v}{\omega(1+y)}\right]\cos[\omega(1+y)t] - \right.$$

$$-\left.\left[\boldsymbol{M}[\alpha_0] - \frac{w}{\omega(1+y)}\right]\sin[\omega(1+y)t]\right\} + \omega\int_0^t F(\tau)\varphi_2(\tau)\sin[\omega(1+y)\tau]\,d\tau +$$

$$+\omega\int_0^t F(\tau)\varphi_1(\tau)\cos[\omega(1+y)\tau]\,d\tau \qquad (3.2)$$

where

$$F(t) = \exp\left[-\tfrac{1}{2}\omega^2 f(t)\right]$$

Turning to Formulas (2.15) we see that for large values of t one can assume the function $f(t)$ to be a linear function of time since

$$f(t) = 2t\int_0^t \boldsymbol{K}_y(\tau)\,d\tau - 2\int_0^t \tau\boldsymbol{K}_y(\tau)\,d\tau \approx 2t\int_0^\infty \boldsymbol{K}_y(\tau)\,d\tau -$$

$$-2\int_0^\infty \tau\boldsymbol{K}_y(\tau)\,d\tau = at - b, \qquad a = 2\pi S_y(0) \qquad (3.3)$$

Here $S_y(\omega)$ is the spectral density of $Y(t)$ and, consequently, the coefficient a cannot be negative and goes to zero only when $S_y(0) = 0$. This will occur, for instance, when $Y(t)$ is a derivative of a stationary random function.

On the other hand, Formulas (2.15) show that

$$\lim\varphi_1(t) = c_1, \qquad \lim\varphi_2(t) = c_2 \quad \text{as} \quad t\to\infty \qquad (3.4)$$

Let $a = 0$. If in this case $c_1 = 0$ and $c_2 = 0$, then from (3.1) and (3.2) it follows that for sufficiently large t the mean position of the pendulum axis will begin to precess with a constant angular velocity $\omega(1 + y)$. The amplitude of this precessional motion depends not only on the mathematical expectation of the initial conditions (first terms in the formulas), but also upon the correlation functions of the coupling between the vertical and horizontal components of acceleration of the support (second terms). Here the axis of precession will not coincide with the vertical, since $M[\alpha(t)]$ and $M[\beta(t)]$ will contain besides the harmonic terms, items of the form

$$\omega \int_0^t [F(\tau)\varphi_j(\tau) - c_j F(\infty)] \begin{matrix}\cos\\ \sin\end{matrix} [\omega(1+y)\tau]\, d\tau \quad (j = 1,\ 2)$$

which can be assumed to be constant for large t.

For $a > 0$ the precession of $M[\alpha(t)]$ and $M[\beta(t)]$, which depends on the initial conditions, will attenuate with time, since the corresponding terms of (3.1) and (3.2) contain the factor $F(t)$ and $\lim f(t) = 0$ as $t \to \infty$. For the same reason the precession caused by the presence of a correlation coupling between the acceleration components will also attenuate.

The attenuation of the precessional motion without damping, which is caused by the presence of the random function $Y(t)$ in the left-hand side of (1.4), distinguishes the studied case from the motion of gyroscopic pendulum without random vertical motions of its point of suspension.

In the presence of damping, as can be seen from Formulas (2.17) and (2.19), the attenuation of the precessional motion will take place also for $S_y(0) \neq 0$. After this attenuation of the precessional motion, $M[\alpha(t)]$ and $M[\beta(t)]$ will be different from zero.

Let us study the general character of the change of the dispersion of the angular deflection of the pendulum axis. Here we shall restrict ourselves to the study of $D[\alpha_1(t)]$ and $D[\beta_1(t)]$, i.e. the dispersion of the deflections which are not related to the initial conditions.

Without damping, Formula (2.23) takes on the form

$$\boldsymbol{M}[|\delta_1(t)|^2] = \omega^2 \int_0^t\int_0^t F(\tau_2 - \tau_1) \Big\{\varphi_1(\tau_1 - \tau_2)\varphi_1(\tau_2 - \tau_1) +$$

$$+ \varphi_2(\tau_1 - \tau_2)\varphi_2(\tau_2 - \tau_1) + \frac{1}{\omega^2}[\boldsymbol{K}_v(\tau_2 - \tau_1) + \boldsymbol{K}_w(\tau_2 - \tau_1)]\Big\} d\tau_1 d\tau_2 \quad (3.5)$$

Under the integral sign we have an even function of the difference $\tau = \tau_2 - \tau_1$. Thus, we can carry out one integration:

$$\boldsymbol{M}[|\delta_1(t)|^2] = 2\omega^2 \int_0^t (t - \tau) F(\tau) \Big\{\varphi_1(\tau)\varphi_1(-\tau) +$$

$$+\varphi_2(\tau)\varphi_2(-\tau)+\frac{1}{\omega^2}[\boldsymbol{K}_v(\tau)+\boldsymbol{K}_w(\tau)]\Big\}\,d\tau \qquad (3.6)$$

For $n = 0$ the Expression (2.30) becomes

$$\operatorname{Re}\boldsymbol{M}[\delta_1^2(\tau)]=\omega^2\int_0^t\int_0^t F^3(\tau_1)F^3(\tau_2)F^{-1}(\tau_2-\tau_1)\Big\{\Phi_{2,2}(\tau_1,\tau_2)-\Phi_{1,1}(\tau_1,\tau_2)+$$
$$+\frac{1}{\omega^2}[\boldsymbol{K}_w(\tau_2-\tau_1)-\boldsymbol{K}_v(\tau_2-\tau_1)]\Big\}\,d\tau_1 d\tau_2 \qquad (3.7)$$

If $a = 0$, $c_1 \neq 0$ and $c_2 \neq 0$, then (3.4) will grow with time, and (3.7) will not contain any terms which grow as $t \to \infty$. Consequently, the dispersions $\alpha_1(t)$ and $\beta_1(t)$ will grow with time.

If $a > 0$, then (3.6) as well as (3.7) tend towards constants with time, and consequently, also the dispersions of the angular deflections of the axis of the gyropendulum will tend towards constants.

In the presence of damping Formulas (2.23) and (2.30) still remain valid. Since in this case, the exponents of $-D(\tau_1, \tau_2)$ and $-D_1(\tau_1, \tau_2)$ tend to $-\infty$ with an increase in their arguments, while all remaining factors in the integrands are either bounded or are decreasing, the dispersions of $\alpha_1(t)$ and $\beta_1(t)$ will tend towards constants as $t \to \infty$ regardless of the value of a.

4. As an example, let us study the behavior of a gyropendulum placed aboard a ship. In contrast to [4] we shall consider here also the vertical component of the acceleration of the pendulum support. The heaving of the ship is assumed to be irregular. Let us assume that the pendulum is placed in a diametral plane of the ship at a distance x_c from the midship bulkhead and that it has an elevation above the center of gravity of the ship (at the equilibrium position of the ship) equal to z_c.

For the sake of simplicity we shall assume that there is no orbital motion of the center of gravity of the ship and no jerking. We shall take into account only the rolling and pitching motions which are characterized by the roll angle $\Theta(t)$ and the pitch angle $\Psi(t)$. In this case, the components of acceleration of the point of suspension of the pendulum are

$$A_\xi=-x_c(\dot\Psi^2+\ddot\Psi\Psi)+z_c\ddot\Psi,\qquad A_\eta=-z_c\ddot\Theta,\qquad A_\zeta=-x_c\ddot\Psi-z_c(\dot\Psi^2+\dot\Theta^2) \qquad (4.1)$$

One can assume with sufficient accuracy that the angles of the ship movement are stationary, not coupled by normal functions [3]. In this case, the correlation functions of the angular velocities $\Omega_1 = \dot{\Theta}$ and $\Omega_1 = \dot{\Psi}$ can be written in the following form

$$\boldsymbol{K}_j(\tau) = \sigma_j^2 e^{-\mu_j|\tau|}\left(\cos\lambda_j\tau + \frac{\mu_j}{\lambda_j}\sin\lambda_j|\tau|\right) \qquad (j = 1, 2) \tag{4.2}$$

where λ_1 and λ_2 are close to the natural frequencies of the roll and pitch motion, respectively; μ_1 and μ_2 depend on the character of the wave motion and the parameters of the ship. For computational purposes we let

$\sigma_1 = \sigma_2 = 0.075$ 1/sec, $\mu_1 = 0.05$ 1/sec, $\mu_2 = 0.10$ 1/sec,
$\lambda_1 = 0.75$ 1/sec, $\lambda_2 = 1.50$ 1/sec, $x_c = 28$ m, $z_c = 14$ m,
$H = 1.708 \times 10^5$ gr cm sec, mgl = 1250 gr cm.

At first, let us retain in (4.1) only terms of the first order. In this case

$$A_\xi = z_c\ddot{\Psi}, \quad A_\eta = -z_c\ddot{\Theta}, \quad A_\zeta = -x_c\ddot{\Psi}, \quad a_\xi = a_\eta = a_\zeta = 0$$

$$Y(t) = -\frac{x_c}{g}\ddot{\Psi}, \quad V(t) = kz_c\ddot{\Theta}, \quad W(t) = kz_c\ddot{\Psi}, \quad v = w = 0 \tag{4.3}$$

which together with (2.15) yields

$$f(t) = \frac{2x_c^2\sigma_2^2}{g^2}\left[1 - e^{-\mu_2|\tau|}\left(\cos\lambda_2\tau + \frac{\mu_2}{\lambda_2}\sin\lambda_2|\tau|\right)\right] \tag{4.4}$$

$$\varphi_1'(t) = 0, \qquad \varphi_2(t) = -\frac{kx_cz_c\sigma_2^2(\mu_2^2 + \lambda_2^2)}{g\lambda_2}e^{-\mu_2|t|}\sin\lambda_2|t|$$

When we assume zero initial conditions and no damping we obtain according to (2.17)

$$\boldsymbol{M}[\alpha(t)] = -\frac{g^2z_c\varkappa(\mu_2^2 + \lambda_2^2)}{\lambda_2x_c}e^{-\varkappa}\int_0^t\exp\left[-\mu_2\tau + \varkappa e^{-\mu_2\tau}\left(\cos\lambda_2\tau + \frac{\mu_2}{\lambda_2}\sin\lambda_2\tau\right)\right]\times$$
$$\times\sin\lambda_2\tau\cos\omega\tau d\tau \tag{4.5}$$

$$\boldsymbol{M}[\beta(t)] = \frac{g^2z_c\varkappa(\mu_2^2 + \lambda_2^2)}{\lambda_2x_c}e^{-\varkappa}\int_0^t\exp\left[-\mu_2\tau + \varkappa e^{-\mu_2\tau}\left(\cos\lambda_2\tau + \frac{\mu_2}{\lambda_2}\sin\lambda_2\tau\right)\right]\times$$
$$\times\sin\lambda_2\tau\sin\omega\tau d\tau$$

where

$$\varkappa = k^2x_c^2\sigma_2^2$$

For the chosen numerical values $\kappa = 1.25 \times 10^{-6}$. Then one can expand the exponential functions under the integral sign in (4.5) in series of powers of κ and retain only the first term of the expansion.

After the integration is carried out, terms containing the factor $\exp(-\mu_2 t)$ are neglected, and it is assumed that for the chosen numerical values $\omega \ll \lambda_2$ and $\omega \ll \mu_2$, we obtain

$$M[\alpha(t)] \approx -\varkappa \frac{z_c}{x_c} \quad \text{or} \quad M[\alpha(t)] \approx -6.2\cdot 10^{-7} = -0.13''$$

In the given case, disregarding the fact that $a = 0$, a regular precession does not occur, since $c_1 = c_2 = 0$, because of the form of the relation between A_ξ, A_η and A_ζ.

When we substitute (4.4) into (3.6) and utilize the same approximations we obtain

$$M[|\gamma(t)|^2] = M[|\delta_1(t)|^2] \approx \frac{\varkappa^2 z_c^2(\mu_2^2+\lambda_2^2)}{2\mu_2 x_c^4}\, t + 2\varkappa \frac{z_c^2}{x_c^2}\left(1+\frac{\sigma_1^2}{\sigma_2^2}\right)$$

Substituting the numerical values we obtain

$$M[|\gamma(t)|^2] = 4.4\cdot 10^{-12}\, t + 1.25\cdot 10^{-6}$$

When we compute Re $M[\delta_1^2(t)]$ in a similar fashion we find that Re $M[\delta_1^2(t)] \ll M[|\delta_1(t)|^2]$ and since $M[\beta(t)] = 0$ and $M[\alpha(t)]$ is small, we obtain for the mean square deflections of the angles $\alpha(t)$ and $\beta(t)$

$$\sigma_\alpha \approx \sigma_\beta \approx \frac{1}{\sqrt{2}}\sqrt{M[|\gamma_1(t)|^2]} = 10^{-3}\sqrt{0.62 + 2.2\cdot 10^{-6} t}$$

which yields, for instance, for t = 10 min and t = 24 hr, $\sigma_\alpha \approx \sigma_\beta \approx 2.7'$ and $\sigma_\alpha \approx \sigma_\beta \approx 3.1'$, respectively.

In the presence of damping, numerical results can also be obtained quite simply if n is assumed to be small, and one performs series expansions in terms of κ and n in the formulas for the mathematical expectations and dispersions.

If we retain the discarded second order terms in the original expressions (4.1), then the functions $Y(t)$, $V(t)$ and $W(t)$ will already be not normal, and all formulas obtained for the moments of the random quantities $\alpha(t)$ and $\beta(t)$, lack accuracy, since in their derivation we used the characteristic functions for normal random quantities. An improved accuracy can be achieved using the expression for the characteristic function which corresponds to the expansion of the propagation law into an Edgewart series [2], which takes into account higher moments of random quantities. In the given example, the deviation from the normal law is caused by second order terms and is not significant. Thus, assuming that all formulas derived above are also applicable in the present case, we shall consider only changes of the values of the mathematical expectations and dispersions $Y(t)$, $V(t)$ and $W(t)$.

If we keep the nonlinear terms in (4.1) we obtain

$$y = -\frac{z_c}{g}(\sigma_1^2 + \sigma_2^2), \qquad v = w = 0 \qquad \boldsymbol{K}_y(\tau) = -\frac{x_c^2}{g^2}\ddot{\boldsymbol{K}}_2(\tau) + \frac{2z_c^2}{g^2}[\boldsymbol{K}_2^2(\tau) + \boldsymbol{K}_1^2(\tau)]$$

$$\boldsymbol{K}_v(\tau) = -z_c^2\ddot{\boldsymbol{K}}_1(\tau), \qquad \boldsymbol{K}_w(\tau) = -k^2 z_c^2 \boldsymbol{K}_2(\tau) + 2kx_c^2 \frac{d^4}{d\tau^4}\boldsymbol{K}_\psi^2(\tau)$$

$$f(\tau) = \frac{2x_c^2}{g^2}[\boldsymbol{K}_2(0) - \boldsymbol{K}_2(\tau)] + \frac{4z_c^2}{g^2}\int_0^t (t-\tau)[\boldsymbol{K}_1^2(\tau) + \boldsymbol{K}_2^2(\tau)]\,d\tau$$

$$R_{vy}(\tau) = 0, \qquad R_{wy}(\tau) = \frac{ky_c z_c}{g}\ddot{\boldsymbol{K}}_2(\tau) + \frac{2kx_c z_c}{g}\frac{d}{d\tau}\left[\dot{\boldsymbol{K}}_\psi(\tau)\ddot{\boldsymbol{K}}_\psi\omega\right]$$

As a result of the presence of the second term in the expression for $K_y(\tau)$ the function $f(t)$ will depend on time for large t; nevertheless, although the correlation functions of the coupling between $R_{vy}(\tau)$ and $R_{wy}(\tau)$ did change the constants c_1 and c_2, will still be equal to zero as before. Thus, the general character of the motion of the pendulum will remain as before, only that, because of the fact that $a > 0$, the attenuation of the precessional motion will proceed more rapidly, and $D[|\gamma(t)|]$ will tend towards a constant value.

As was noted above, the consideration in the expressions of characteristic functions of the higher moments of the ordinates of functions $Y(t)$, $V(t)$ and $W(t)$ does not present principal difficulties, however, it naturally complicates the computational process.

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